

LIKEWISE THETA FUNCTIONS OF RANK r ON \mathbb{R}^d : ANALYTIC PROPERTIES AND ASSOCIATED SEGAL-BARGMANN TRANSFORM

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ABSTRACT. We introduce and study the Hilbert space of (L^2, Γ, χ) -likewise theta functions on \mathbb{R}^d with respect to a given discrete subgroup Γ of arbitrary rank and a character χ of Γ . A concrete description is given and an orthonormal basis is then constructed. Its range by the classical Segal-Bargmann transform is also characterized and leads to the so-called theta-Bargmann Fock space.

1. INTRODUCTION

Analytic properties of the holomorphic automorphic functions, associated with a full-rank lattice Γ in the d -dimensional complex space \mathbb{C}^d and a given mapping χ on Γ with values in unit circle of \mathbb{C} , are well studied in the literature. Such functions play important roles in number theory and abelian varieties [11, 17, 4, 14], cryptography and coding theory [16, 18], as well as in quantum field theory [5]. Extending these properties to the case of an arbitrary rank discrete subgroup is an interesting area of research. The more recent investigation in this context has been discussed in [8, 19] for rank one discrete subgroups and next generalized in [9] to isotropic discrete subgroups of rank less or equal to d . The elaboration of these properties lies in the holomorphic character of the considered functions attached to the complex structure of \mathbb{C}^d . This tool is lost when working on \mathbb{R}^d instead of $\mathbb{R}^{2d} = \mathbb{C}^d$, where d is not necessary even. Thus, inspired by the impact of Segal-Bargmann transform on signal processing and time-frequency analysis on the free Hilbert space $L^2(\mathbb{R}^d)$ (see for example [10]) and motivated by the fact that many signals in practice are quasi-periodic, we will develop and investigate in a natural way a parallel theory for the space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ of (L^2, Γ, χ) -likewise theta functions associated to an arbitrary discrete subgroup of rank r in \mathbb{R}^d . For the full-rank lattice Γ , the Segal-Bargmann transform of (L^2, Γ) -periodic functions is characterized to be the space of L^2 -Bloch wave functions [3].

The aim of the present paper is then two folds. Firstly, we give explicit description of the elements of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ and next construct an explicit orthonormal basis in terms of the modified Fourier expansion and Hermite polynomials. Secondly, we consider the Segal-Bargmann transform and prove that it maps isometrically the space of (L^2, Γ, χ) -likewise theta functions on \mathbb{R}^d onto the well-studied space of $(L^2, \tilde{\Gamma}, \tilde{\chi})$ -holomorphic theta functions on \mathbb{C}^d considered in [9] for a special pair $(\tilde{\Gamma}; \tilde{\chi})$. This gives rise to the introduction of the so-called theta-Segal-Bargmann transform involving an integral representation over a fundamental domain with kernel function given in terms of the multidimensional Reimann theta function with special characteristics.

This paper is organized as follows. Section 2 is devoted to the exact statement of our main results. In Section 3, we establish some basic properties of the space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ of the (L^2, Γ, χ) -likewise theta functions on \mathbb{R}^d and give the proof of the Theorem 2.1. In Section

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4, we review the needed properties concerning the space $\mathcal{F}_{\tilde{\Gamma}, \tilde{\chi}}^{2, \nu}(\mathbb{C}^d)$ of $(L^2, \tilde{\Gamma}, \tilde{\chi})$ -holomorphic theta functions considered in [9]. In Section 5, we prove Theorem 2.2 concerning to the characterization of the range of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ by the Segal-Bargmann transform as well as Theorem 2.3 leading to the notion of theta-Segal-Bargmann transform.

2. STATEMENT OF MAIN RESULTS

In order to give a concise picture of our results, we endow the d -dimensional Euclidean space \mathbb{R}^d with the usual scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $\Gamma = \Gamma_r$ be a discrete subgroup of rank r ; $r = 0, 1, \dots, d$, in the additive group $(\mathbb{R}^d, +)$. By \mathbb{R}^d / Γ we denote the associated abelian orbital group equipped with the quotient topology and the Haar measure. Associated with the data of Γ , a nonnegative real number ν and a given mapping χ with values in the unit circle of \mathbb{C} on Γ , we consider the functional space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ of Borel measurable functions f on \mathbb{R}^d satisfying the functional equation

$$f(x + \gamma) = \chi(\gamma) e^{\nu \langle x + \frac{\gamma}{2}, \gamma \rangle} f(x), \quad (2.1)$$

for almost every $x \in \mathbb{R}^d$ and every $\gamma \in \Gamma$, and

$$\|f\|_{\Gamma, \nu}^2 := \int_{\Lambda(\Gamma)} |f(x)|^2 e^{-\nu \|x\|^2} d\lambda(x) < \infty. \quad (2.2)$$

Here $\Lambda(\Gamma)$ is a fundamental domain of Γ in \mathbb{R}^d representing \mathbb{R}^d / Γ and gives rise to a compact fundamental domain of Γ in $\mathbb{V}_\Gamma = \text{Span}_{\mathbb{R}}(\Gamma)$, the r -dimensional real vector space generated by Γ . Notice that the quantity $\| \cdot \|_{\Gamma, \nu}$ makes sense and it is independent of the choice of the fundamental domain, for the function $x \mapsto |f(x)|^2 e^{-\nu \|x\|^2}$ being a Γ -periodic on \mathbb{R}^d for every given f satisfying (2.1). Furthermore, $\| \cdot \|_{\Gamma, \nu}$ defines a norm on $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. The corresponding scalar product is given by

$$\langle f, g \rangle_{\Gamma, \nu} := \int_{\Lambda(\Gamma)} f(x) \overline{g(x)} e^{-\nu \|x\|^2} d\lambda(x). \quad (2.3)$$

We will show that the functional space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ is nontrivial if and only if χ is a character. In this case χ has a representation of the form $\chi(\gamma) = e^{2i\pi \langle \gamma, v_\chi \rangle}$ (see Lemma 3.5). Under such condition, we give a concrete description of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ using Fourier analysis related to the dual lattice Γ^* of Γ . Namely, by identifying \mathbb{R}^d to $\mathbb{V}_\Gamma \times \mathbb{V}_\Gamma^{\perp(\cdot)}$, where $\mathbb{V}_\Gamma^{\perp(\cdot)}$ is the complement orthogonal of \mathbb{V}_Γ in \mathbb{R}^d with respect to $\langle \cdot, \cdot \rangle$, and denoting by $\mathbf{H}_\mathbf{k}^\nu$ the $(d - r)$ -dimensional Hermite polynomials, we can assert the following.

Theorem 2.1. *The family of functions on $\mathbb{V}_\Gamma \times \mathbb{V}_\Gamma^{\perp(\cdot)}$ defined by*

$$e_{\gamma^*, \mathbf{k}}(x) = e_{\gamma^*, \mathbf{k}}(x_1, x_2) := e^{\frac{\nu}{2} \langle x_1, x_1 \rangle + 2\pi i \langle v_\chi + \gamma^*, x_1 \rangle} \mathbf{H}_\mathbf{k}^\nu(x_2) \quad (2.4)$$

for varying $\gamma^ \in \Gamma^*$ of Γ and $\mathbf{k} = (k_1, \dots, k_{d-r}) \in (\mathbb{Z}^+)^{d-r}$, constitutes an orthogonal basis of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ with*

$$\|e_{\gamma^*, \mathbf{k}}\|_{\Gamma, \nu}^2 = \text{vol}(\Lambda_1(\Gamma)) \left(\frac{\pi}{\nu} \right)^{(d-r)/2} 2^{|\mathbf{k}|} \mathbf{k}!. \quad (2.5)$$

The characterization of the range of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ by the Segal-Bargmann transform,

$$[\mathcal{B}\varphi](z) = \left(\frac{\nu}{\pi} \right)^{\frac{3d}{4}} \int_{\mathbb{R}^d} e^{\sqrt{2}\nu \langle z, x \rangle - \frac{\nu}{2} \langle z, z \rangle} \varphi(x) e^{-\nu \|x\|^2} d\lambda(x); \quad z \in \mathbb{C}^d, \quad (2.6)$$

involves the shifted lattice $\tilde{\Gamma} := \Gamma / \sqrt{2}$ and the $\tilde{\Gamma}$ -character defined by $\tilde{\chi}(\tilde{\gamma}) = e^{2i\pi \langle \tilde{\gamma}, \sqrt{2}v_\chi \rangle}$. The natural extension of $\langle \cdot, \cdot \rangle$ on \mathbb{R}^d to $\mathbb{C}^d = \mathbb{V}_\mathbb{C} \oplus \mathbb{V}_\mathbb{C}^{\perp_H}$ with $\mathbb{V}_\mathbb{C} = \mathbb{V}_\Gamma + i\mathbb{V}_\Gamma$ is also denoted by $\langle \cdot, \cdot \rangle$.

Theorem 2.2. *The Segal-Bargmann transform \mathcal{B} given through (2.6) defines an isometric isomorphism from $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ onto $\mathcal{F}_{\tilde{\Gamma},\tilde{\chi}}^{2,\nu}(\mathbb{C}^d)$.*

The last theorem of this paper gives another integral representation for \mathcal{B} when restricted to $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$. This representation is a coherent state transform and encodes the Hilbert structures of $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ and $\mathcal{F}_{\tilde{\Gamma},\tilde{\chi}}^{2,\nu}(\mathbb{C}^d)$. In fact, the kernel function is the bilateral generating function making appeal of both bases. Moreover, it can be expressed in terms of the multidimensional Riemann theta function $\Theta_{\alpha,\beta}(z|\Omega)$ with special characteristics. To this end, recall that ([15, 13]):

$$\Theta_{\alpha,\beta}(z|\Omega) = \sum_{n \in \mathbb{Z}^r} e^{2i\pi \left\{ \frac{1}{2}(\alpha+n)\Omega(\alpha+n) + (\alpha+n)(z+\beta) \right\}} \quad (2.7)$$

where $\alpha, \beta \in \mathbb{R}^r$ and Ω a symmetric matrix in $\mathbb{C}^{r \times r}$ with strictly positive definite imaginary part. The positive definiteness of $\Im(\Omega)$ guarantees the convergence of (2.7) on \mathbb{C}^r .

Theorem 2.3. *The Segal-Bargmann transform \mathcal{B} on $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ is also given by*

$$[\mathcal{B}\varphi](z) = \int_{\Lambda(\Gamma)} A_{\Gamma,\chi}^{\nu}(z; x) \varphi(x) e^{-\nu\|x\|^2} d\lambda(x); \quad z \in \mathbb{C}^d, \quad (2.8)$$

where the kernel function $A_{\Gamma,\chi}^{\nu}(z; x)$ is given in terms of the modified theta function as follows

$$A_{\Gamma,\chi}^{\nu}(z; x) = \left(\frac{\nu}{\pi} \right)^{\frac{3d}{4}} e^{\sqrt{2}\nu\langle z, x \rangle - \frac{\nu}{2}\langle z, z \rangle} \Theta_{0, G\beta_{\chi}} \left(\frac{iv}{2\pi} G(x_1 - \sqrt{2}z_1) \middle| \frac{iv}{2\pi} G \right). \quad (2.9)$$

Here $x = x_1 + x_2 \in \mathbb{V}_{\Gamma} \oplus \mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$, $z = z_1 + z_2 \in \mathbb{V}_{\mathbb{C}} \oplus \mathbb{V}_{\mathbb{C}}^{\perp H}$, $G := (\langle \omega_i, \omega_j \rangle)_{1 \leq i, j \leq r}$ is the Gram-Schmidt matrix of the form $\langle \cdot, \cdot \rangle$ restricted to the vector space \mathbb{V}_{Γ} generated by a basis $\{\omega_i; i = 1, 2, \dots, r\}$ of Γ , and $\beta_{\chi} = (\beta_1, \dots, \beta_r) \in \mathbb{R}^r$ are the coordinates of v_{χ} .

3. BASIC PROPERTIES OF $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ AND PROOF OF THEOREM 2.1

Keep notations as above and provide the following definition.

Definition 3.1. The space $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ will be called the space of (L^2, Γ, χ) -likewise theta functions on \mathbb{R}^d attached to the discrete subgroup Γ . Similarly, we define $\mathcal{C}_{\Gamma,\chi}^{\nu,\infty}(\mathbb{R}^d)$ to be the space of $(\mathcal{C}^{\infty}, \Gamma, \chi)$ -likewise theta functions, that is the space of all \mathcal{C}^{∞} -complex-valued functions f on \mathbb{R}^d satisfying

$$f(x + \gamma) = \chi(\gamma) e^{\nu \langle x + \frac{\gamma}{2}, \gamma \rangle} f(x), \quad (3.1)$$

for every $x \in \mathbb{R}^d$ and $\gamma \in \Gamma$.

The existence of such spaces is encoded in the data (ν, Γ, χ) . In fact, the following result gives a necessary and sufficient condition to $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ and $\mathcal{C}_{\Gamma,\chi}^{\nu,\infty}(\mathbb{R}^d)$ be nontrivial.

Proposition 3.2. *The vector space $L_{\Gamma,\chi}^{2,\nu}(\mathbb{R}^d)$ (resp. $\mathcal{C}_{\Gamma,\chi}^{\nu,\infty}(\mathbb{R}^d)$) is nonzero if and only if χ is a character on Γ , i.e., for all $\gamma, \gamma' \in \Gamma$, we have*

$$\chi(\gamma + \gamma') = \chi(\gamma)\chi(\gamma'). \quad (3.2)$$

Proof. For the necessary condition with $\mathcal{C}_{\Gamma,\chi}^{\nu,\infty}(\mathbb{R}^d)$, assume that $\mathcal{C}_{\Gamma,\chi}^{\nu,\infty}(\mathbb{R}^d)$ is nontrivial and let f be a nonzero function belonging to $\mathcal{C}_{\Gamma,\chi}^{\nu,\infty}(\mathbb{R}^d)$. Hence, for every $\gamma, \gamma' \in \Gamma$ and $x \in \mathbb{R}^d$, we can write $f(x + \gamma + \gamma')$ in the following forms

$$f(x + \gamma + \gamma') = f((x + \gamma) + \gamma') = \chi(\gamma')\chi(\gamma) e^{\nu \langle x + \frac{\gamma + \gamma'}{2}, \gamma + \gamma' \rangle} f(x) \quad (3.3)$$

and

$$f(x + \gamma + \gamma') = f(x + (\gamma + \gamma')) = \chi(\gamma' + \gamma) e^{\nu \langle x + \frac{\gamma + \gamma'}{2}, \gamma + \gamma' \rangle} f(x). \quad (3.4)$$

Hence the desired result follows by equating the right hand-sides of (3.3) and (3.4) and using the fact that $f(x_0) \neq 0$ for certain x_0 . The necessary condition concerning $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ can be handled in a similar way with additional consideration. In fact, the x_0 is taken in $\mathbb{R}^d \setminus \widetilde{\mathcal{D}}_f$, where $\widetilde{\mathcal{D}}_f = \cup_{\gamma \in \Gamma} (D_f + \gamma)$ and D_f is a negligible subset of \mathbb{R}^d such that f is well defined on its complementary $\mathbb{R}^d \setminus D_f$. Notice that $\widetilde{\mathcal{D}}_f$ is also negligible since Γ is a countable set.

The proof of the sufficient condition follows by considering the Poincaré series

$$[\mathcal{P}_{\Gamma, \chi} \psi](x) = \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)} e^{-\nu \langle x + \frac{\gamma}{2}, \gamma \rangle} \psi(x + \gamma) \quad (3.5)$$

related to given \mathcal{C}^∞ complex-valued function ψ with compact support contained in the interior of $\Lambda(\Gamma)$ (such ψ exists from the classical analysis). The series (3.5) is then well defined, since every $x \in \mathbb{R}^d$ has a unique representation in $\Lambda(\Gamma)$. Moreover, $\mathcal{P}_{\Gamma, \chi} \psi$ is a nonzero function belonging to $\mathcal{C}_{\Gamma, \chi}^{v, \infty}(\mathbb{R}^d) \cap L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. Indeed, we have $[\mathcal{P}_{\Gamma, \chi} \psi] = \psi \neq 0$ on $\Lambda(\Gamma)$, since $\chi(0) = 1$ and $x + \gamma \notin \text{Supp}(\psi)$ for all $x \in \Lambda(\Gamma)$. Now, making use of the fact that $\chi(\gamma'' - \gamma) = \chi(\gamma'') \overline{\chi(\gamma)}$, for χ being a character, combined with the symmetry of the bilinear form $\langle \cdot, \cdot \rangle$, we obtain

$$[\mathcal{P}_{\Gamma, \chi} \psi](x + \gamma) = \chi(\gamma) e^{\nu \langle x + \frac{\gamma}{2}, \gamma \rangle} [\mathcal{P}_{\Gamma, \chi} \psi](x)$$

for every $\gamma \in \Gamma$ and $x \in \mathbb{R}^d$. This completes the proof. \square

Remark 3.3. The proof of "if" in the previous theorem can be reworded if we are able to exhibit an explicit nonzero complex-valued function belonging to $\mathcal{C}_{\Gamma, \chi}^{v, \infty}(\mathbb{R}^d) \cap L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. This is contained in Lemma 3.5 below.

Remark 3.4. Notice that $\mathcal{C}_{\Gamma, \chi}^{v, \infty}(\mathbb{R}^d)$ is a prehilbertian space when equipped with the scalar product (2.3). Thus, by functional analysis theory it has a completion. It should be proved later that $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ is in fact the completion of $\mathcal{C}_{\Gamma, \chi}^{v, \infty}(\mathbb{R}^d)$ (Proposition 3.9).

Now, we deal with $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ under the assumption that χ is a character. The discrete subgroup Γ of $(\mathbb{R}^d, +)$ can be viewed as a \mathbb{Z} -module of dimension r . Therefore, Γ take the following form

$$\Gamma = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_r$$

for some \mathbb{R} -linearly independent vectors $\omega_1, \dots, \omega_r \in \mathbb{R}^d$. A corresponding fundamental domain is proved to be given by

$$\Lambda(\Gamma) \simeq \mathbb{R}^d / \Gamma \simeq \Lambda_1(\Gamma) \times \mathbb{V}_\Gamma^{\perp \langle \cdot, \cdot \rangle},$$

where $\Lambda_1(\Gamma)$ is a compact fundamental domain of Γ in the r -dimensional real vector space $\mathbb{V}_\Gamma = \text{Span}_{\mathbb{R}}(\Gamma)$, constituting of all \mathbb{R} -linearly finite combinations of elements of Γ . By $\mathbb{V}_\Gamma^{\perp \langle \cdot, \cdot \rangle}$ we denote its complement orthogonal with respect to $\langle \cdot, \cdot \rangle$. Accordingly, we can split \mathbb{R}^d as

$$\mathbb{R}^d = \mathbb{V}_\Gamma \oplus \mathbb{V}_\Gamma^{\perp \langle \cdot, \cdot \rangle} \quad (3.6)$$

and the symmetric bilinear form $\langle \cdot, \cdot \rangle$ as

$$\langle x, y \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = Q_1(x_1, y_1) + Q_2(x_2, y_2), \quad (3.7)$$

for every $x = x_1 + x_2$ and $y = y_1 + y_2$ in \mathbb{R}^d with $x_1, y_1 \in \mathbb{V}_\Gamma$ and $x_2, y_2 \in \mathbb{V}_\Gamma^{\perp \langle \cdot, \cdot \rangle}$, where Q_1 (resp. Q_2) is the restriction of $\langle \cdot, \cdot \rangle$ to \mathbb{V}_Γ (resp. $\mathbb{V}_\Gamma^{\perp \langle \cdot, \cdot \rangle}$) and can be viewed as a positive definite symmetric bilinear form on \mathbb{V}_Γ (resp. $\mathbb{V}_\Gamma^{\perp \langle \cdot, \cdot \rangle}$).

Now in view of (3.7), the functional equation (2.1) reduces further to

$$f(x_1 + \gamma, x_2) = \chi(\gamma) e^{\nu \langle x_1 + \frac{\gamma}{2}, \gamma \rangle} f(x_1, x_2) \quad (3.8)$$

with $f(x_1, x_2) := f(x_1 + x_2)$ for all $x_1 \in \mathbb{V}_\Gamma$, $x_2 \in \mathbb{V}_\Gamma^{\perp(\cdot, \cdot)}$ and $\gamma \in \Gamma$.

Lemma 3.5. *Under the assumption that χ is a character, we have the following*

- i) *There exists a vector $v_\chi \in \mathbb{V}_\Gamma$ (depending in the choice of the basis of Γ) such that for all $\gamma \in \Gamma$, we have*

$$\chi(\gamma) = e^{2i\pi \langle v_\chi, \gamma \rangle}.$$

- ii) *The function*

$$x \mapsto \psi_{v_\chi}(x) := e^{\frac{\nu}{2} \langle x_1, x_1 \rangle + 2i\pi \langle x_1, v_\chi \rangle} \quad (3.9)$$

belongs to $\mathcal{C}_{\Gamma, \chi}^{\nu, \infty}(\mathbb{R}^d) \cap L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$.

Proof. Since χ is a Γ -character and its restriction to any $\mathbb{Z}\omega_j$; $1 \leq j \leq r$, is a \mathbb{Z} -character, there exists $\alpha_j \in \mathbb{R}$ such that $\chi(\omega_j) = e^{2i\pi \alpha_j}$. Therefore,

$$\chi(\gamma) = \chi(m_1 \omega_1 + \cdots + m_r \omega_r) = e^{2i\pi(\alpha_1 m_1 + \cdots + \alpha_r m_r)}.$$

Let α_χ stands for $\alpha_\chi = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$ and $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$. Then, we can rewrite $\alpha_1 m_1 + \cdots + \alpha_r m_r$ as

$$\alpha_1 m_1 + \cdots + \alpha_r m_r = {}^t m G (G^{-1} \alpha_\chi) = {}^t m G \beta_\chi = \langle v_\chi, \gamma \rangle,$$

where $G := (\langle \omega_i, \omega_j \rangle)_{1 \leq i, j \leq r}$ is the invertible Gram-Schmidt matrix of Q_1 on the vector space \mathbb{V}_Γ and v_χ is the vector in \mathbb{V}_Γ given by $v_\chi = \beta_1 \omega_1 + \cdots + \beta_r \omega_r$ with $\beta_\chi := (\beta_1, \dots, \beta_r) := G^{-1} \alpha_\chi$. This completes the proof of i).

The proof of ii) can be handled easily using the bilinearity and the symmetry of $\langle \cdot, \cdot \rangle$, keeping in mind that $\chi(\gamma) = e^{2i\pi \langle v_\chi, \gamma \rangle}$. Indeed, we have

$$\psi_{v_\chi}(x + \gamma) = e^{\frac{\nu}{2} \langle x_1 + \gamma, x_1 + \gamma \rangle + 2i\pi \langle x_1 + \gamma, v_\chi \rangle} = \chi(\gamma) \psi_{v_\chi}(x) e^{\nu \langle x + \frac{\gamma}{2}, \gamma \rangle}.$$

The last equality follows since $\langle y_1, \gamma \rangle = \langle y, \gamma \rangle$ for every $y = y_1 + y_2 \in \mathbb{R}^d$ with $(y_1, y_2) \in \mathbb{V}_\Gamma \times \mathbb{V}_\Gamma^{\perp(\cdot, \cdot)}$ and $\gamma \in \Gamma \subset \mathbb{V}_\Gamma$. \square

The existence and the explicit expression of ψ_{v_χ} will play a crucial role in establishing the following result.

Theorem 3.6. *Keep notations as above. Then, $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ is a Hilbert space. Moreover, a function f belongs to $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ if and only if it can be expanded in series as*

$$f(x) = e^{\frac{\nu}{2} \langle x_1, x_1 \rangle} \sum_{\gamma^* \in \Gamma^*} a_{\gamma^*}(x_2) e^{2i\pi \langle v_\chi + \gamma^*, x_1 \rangle}, \quad (3.10)$$

where $\Gamma^* := \{\gamma^* \in \mathbb{V}_\Gamma; \langle \gamma^*, \gamma \rangle \in \mathbb{Z}; \gamma \in \Gamma\}$ denotes the dual of Γ . The involved coefficients $a_{\gamma^*}(x_2)$; $x_2 \in \mathbb{V}_\Gamma^{\perp(\cdot, \cdot)}$, satisfy the growth condition

$$\sum_{\gamma^* \in \Gamma^*} \|a_{\gamma^*}\|_{L^2(\mathbb{V}_\Gamma^{\perp(\cdot, \cdot)}, e^{-\nu \langle x_2, x_2 \rangle} d\lambda(x_2))}^2 < +\infty. \quad (3.11)$$

Proof. The first assertion is obvious. Indeed, for any Cauchy sequence $(f_n)_n$ in $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$, the $(f_n|_{\Lambda(\Gamma)})_n$ is a Cauchy sequence in the Hilbert space $L^2(\Lambda(\Gamma); e^{-\|\xi\|^2} d\lambda)$ and hence converges to some function f defined on $\Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is an arbitrary fundamental domain, it follows that f is defined on \mathbb{R}^d and satisfies (2.1). For the second assertion, let $f \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. Then, the function

$$\varphi(x) := e^{-\frac{\nu}{2} \langle x_1, x_1 \rangle - 2i\pi \langle v_\chi, x_1 \rangle} f(x)$$

is a Γ -periodic function on \mathbb{R}^d in the x_1 -direction. Furthermore, by means of Fubini's theorem, we get

$$\|f\|_{\Gamma, \nu}^2 = \int_{\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}} \left(\int_{\Lambda_1(\Gamma)} |\varphi(x_1, x_2)|^2 d\lambda(x_1) \right) e^{-\nu \langle x_2, x_2 \rangle} d\lambda(x_2) < +\infty. \quad (3.12)$$

Subsequently,

$$\int_{\Lambda_1(\Gamma)} |\varphi(x_1, x_2)|^2 d\lambda(x_1) < +\infty$$

almost everywhere on $\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$, and the function $x_1 \mapsto \varphi(x_1, x_2)$ can be expanded as

$$\varphi(x_1, x_2) = \sum_{\gamma^* \in \Gamma^*} a_{\gamma^*}(x_2) e^{2\pi i \langle x_1, \gamma^* \rangle}, \quad (3.13)$$

where the series converges absolutely and uniformly on $\Lambda_1(\Gamma)$, for almost every fixed $x_2 \in \mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$. Here $a_{\gamma^*}(x_2)$ are the Fourier coefficients given by ([11, p.44]):

$$a_{\gamma^*}(x_2) = \frac{1}{\text{vol}(\Lambda_1(\Gamma))} \int_{\Lambda_1(\Gamma)} \varphi(x_1, x_2) e^{-2\pi i \langle x_1, \gamma^* \rangle} d\lambda(x_1).$$

To prove the growth condition (3.11), we start from (3.10) and make use again of the Fubini theorem. This entails

$$\begin{aligned} \|f\|_{\Gamma, \nu}^2 &= \int_{\Lambda(\Gamma)} \left| e^{\frac{\nu}{2} \langle x_1, x_1 \rangle} \sum_{\gamma^* \in \Gamma^*} a_{\gamma^*}(x_2) e^{2i\pi \langle v_{\chi} + \gamma^*, x_1 \rangle} \right|^2 e^{-\nu \langle x, x \rangle} d\lambda(x) \\ &= \int_{\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}} e^{-\nu \langle x_2, x_2 \rangle} \left(\int_{\Lambda_1(\Gamma)} \left| \sum_{\gamma^* \in \Gamma^*} a_{\gamma^*}(x_2) e^{2i\pi \langle v_{\chi} + \gamma^*, x_1 \rangle} \right|^2 d\lambda(x_1) \right) d\lambda(x_2). \end{aligned}$$

Next, by Parseval's identity in the Hilbert space $L^2(\Lambda_1(\Gamma); d\lambda(x_1))$, we get

$$\int_{\Lambda_1(\Gamma)} \left| \sum_{\gamma^* \in \Gamma^*} a_{\gamma^*}(x_2) e^{2i\pi \langle \gamma^*, x_1 \rangle} \right|^2 d\lambda(x_1) = \text{vol}(\Lambda_1(\Gamma)) \sum_{\gamma^* \in \Gamma^*} |a_{\gamma^*}(x_2)|^2,$$

for almost every $x_2 \in \mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$. Therefore, the square norm $\|f\|_{\Gamma, \nu}^2$ reduces to

$$\begin{aligned} \|f\|_{\Gamma, \nu}^2 &= \text{vol}(\Lambda_1(\Gamma)) \sum_{\gamma^* \in \Gamma^*} \int_{\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}} e^{-\nu \langle x_2, x_2 \rangle} |a_{\gamma^*}(x_2)|^2 d\lambda(x_2) \\ &= \text{vol}(\Lambda_1(\Gamma)) \sum_{\gamma^* \in \Gamma^*} \|a_{\gamma^*}\|_{L^2(\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}, e^{-\nu \langle x_2, x_2 \rangle} d\lambda(x_2))}^2. \end{aligned} \quad (3.14)$$

This completes the proof. \square

To the exact statement of the main result of this section, notice that the subspace $\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$ is generated by some \mathbb{R} -linearly independent vectors, $\omega_{r+1}, \dots, \omega_d \in \mathbb{R}^d$. Without loss of generality, we can assume that the $\omega_{r+1}, \dots, \omega_d$ are orthonormal with respect to the form Q_2 ; the restriction of the usual scalar product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^d to $\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$ (see for example [20]). That is

$$Q_2(\omega_j, \omega_k) = \delta_{jk}; \quad j, k = r+1, \dots, d.$$

Identifying $x_2 = x_{r+1}\omega_{r+1} + \dots + x_d\omega_d$ in $\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$ to its coordinates (x_{r+1}, \dots, x_d) in \mathbb{R}^{d-r} . Then, an orthogonal basis of $L^2(\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}; e^{-\nu \|x_2\|^2} d\lambda(x_2))$ is

$$\mathbf{H}_{\mathbf{k}}^{\nu}(x_2) = \mathbf{H}_{\mathbf{k}}(\sqrt{\nu}x_2); \quad \mathbf{k} \in (\mathbb{Z}^+)^{d-r}, \quad (3.15)$$

where $\mathbf{H}_{\mathbf{k}}$ denotes the $(d-r)$ -dimensional Hermite polynomials

$$\mathbf{H}_{\mathbf{k}}(\xi) = (-1)^{|\mathbf{k}|} e^{\|\xi\|^2} \frac{\partial^{\mathbf{k}}}{\partial \xi_{r+1}^{k_{r+1}} \dots \partial \xi_d^{k_d}} \left(e^{-\|\xi\|^2} \right); \quad (3.16)$$

with $\mathbf{k} = (k_{r+1}, k_{r+2}, \dots, k_d) \in (\mathbb{Z}^+)^{d-r}$, $|\mathbf{k}| = k_{r+1} + k_{r+2} + \dots + k_d$ and $\xi = (\xi_{r+1}, \xi_{r+2}, \dots, \xi_d) \in \mathbb{R}^{d-r}$. Therefore, by means of (3.11), the Fourier coefficient $x_2 \mapsto a_{\gamma^*}(x_2)$ belongs to the Hilbert space $L^2(\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}; e^{-\nu\|x_2\|^2} d\lambda(x_2))$, and it can be expanded as

$$a_{\gamma^*}(x_2) = \sum_{\mathbf{k} \in (\mathbb{Z}^+)^{d-r}} a_{\gamma^*, \mathbf{k}} \mathbf{H}_{\mathbf{k}}^{\nu}(x_2), \quad (3.17)$$

for some complex numbers $a_{\gamma^*, \mathbf{k}}$. Thus, we assert the following

Theorem 3.7. *A complex-valued function f belongs to the Hilbert space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ if and only if it can be expanded as*

$$f(x) := f(x_1, x_2) = \sum_{\gamma^* \in \Gamma^*, \mathbf{k} \in (\mathbb{Z}^+)^{d-r}} a_{\gamma^*, \mathbf{k}} e^{\frac{\nu}{2}\langle x_1, x_1 \rangle + 2\pi i \langle v_{\chi} + \gamma^*, x_1 \rangle} \mathbf{H}_{\mathbf{k}}^{\nu}(x_2) \quad (3.18)$$

for almost every $(x_1, x_2) \in \mathbb{V}_{\Gamma} \times \mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}$, with

$$\|f\|_{\Gamma, \nu}^2 = \text{vol}(\Lambda_1(\Gamma)) \left(\frac{\pi}{\nu}\right)^{(d-r)/2} \sum_{\gamma^* \in \Gamma^*, \mathbf{k} \in (\mathbb{Z}^+)^{d-r}} 2^{|\mathbf{k}|} \mathbf{k}! |a_{\gamma^*, \mathbf{k}}|^2 < +\infty.$$

Remark 3.8. The series in (3.18) converges in the Hilbert space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$.

Proof. The result is a consequence of Theorem 3.6 and the fact that the Fourier coefficients $a_{\gamma^*}(x_2)$ are given by (3.17). More exactly, using the orthogonality of the Hermite polynomials in $L^2(\mathbb{R}^{d-r}, e^{-\|\xi\|^2} d\lambda(\xi))$, we get

$$\begin{aligned} \|a_{\gamma^*}\|_{L^2(\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}, e^{-\nu\langle x_2, x_2 \rangle} d\lambda)}^2 &= \int_{\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}} e^{-\nu\langle x_2, x_2 \rangle} |a_{\gamma^*}(x_2)|^2 d\lambda(x_2) \\ &= \frac{1}{\nu^{(d-r)/2}} \sum_{\mathbf{k} \in (\mathbb{Z}^+)^{d-r}} |a_{\gamma^*, \mathbf{k}}|^2 \int_{\mathbb{R}^{d-r}} |\mathbf{H}_{\mathbf{k}}(\xi)|^2 e^{-\|\xi\|^2} d\lambda(\xi) \\ &= \left(\frac{\pi}{\nu}\right)^{(d-r)/2} \sum_{\mathbf{k} \in (\mathbb{Z}^+)^{d-r}} 2^{|\mathbf{k}|} \mathbf{k}! |a_{\gamma^*, \mathbf{k}}|^2. \end{aligned}$$

□

Now, we are able to give a proof of Theorem 2.1 saying that the family of functions

$$e_{\gamma^*, \mathbf{k}}(x) = e^{\frac{\nu}{2}\langle x_1, x_1 \rangle + 2\pi i \langle v_{\chi} + \gamma^*, x_1 \rangle} \mathbf{H}_{\mathbf{k}}^{\nu}(x_2), \quad (3.19)$$

for $\gamma^* \in \Gamma^*$ and $\mathbf{k} = (k_1, \dots, k_{d-r}) \in (\mathbb{Z}^+)^{d-r}$, constitutes an orthogonal basis of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ with

$$\|e_{\gamma^*, \mathbf{k}}\|_{\Gamma, \nu}^2 = \text{vol}(\Lambda_1(\Gamma)) \left(\frac{\pi}{\nu}\right)^{(d-r)/2} 2^{|\mathbf{k}|} \mathbf{k}!. \quad (3.20)$$

Proof of Theorem 2.1. The orthogonality of the functions $e_{\gamma^*, \mathbf{k}}$; for $\gamma^* \in \Gamma^*$, $\mathbf{k} = (k_1, \dots, k_{d-r}) \in (\mathbb{Z}^+)^{d-r}$, follows from the orthogonality of the Hermite polynomials in $L^2(\mathbb{R}^{d-r}, e^{-\|\xi\|^2} d\lambda)$ and the use of the following well-established identity

$$\int_{\Lambda_1(\Gamma)} e^{2\pi i \langle x_1, \gamma_1^* - \gamma_2^* \rangle} d\lambda(x_1) = \text{vol}(\Lambda_1(\Gamma)) \prod_{j=1}^r \left(\int_0^1 e^{2\pi i t_j \langle \omega_j, \gamma_1^* - \gamma_2^* \rangle} dt_j \right) = \text{vol}(\Lambda_1(\Gamma)) \delta_{\gamma_1^*, \gamma_2^*}$$

for $x_1 = t_1 \omega_1 + \dots + t_r \omega_r \in \Lambda_1(\Gamma)$ with $t_j \in [0, 1]$. In fact, we obtain

$$\begin{aligned} \langle e_{\gamma^*, \mathbf{k}}, e_{\gamma'^*, \mathbf{k}'} \rangle_{\Gamma, \nu} &= \left(\int_{\Lambda_1(\Gamma)} e^{2\pi i \langle x_1, \gamma_1^* - \gamma_2^* \rangle} d\lambda(x_1) \right) \times \left(\int_{\mathbb{V}_{\Gamma}^{\perp(\cdot, \cdot)}} \mathbf{H}_{\mathbf{k}}^{\nu}(x_2) \overline{\mathbf{H}_{\mathbf{k}'}^{\nu}(x_2)} e^{-\nu\langle x_2, x_2 \rangle} d\lambda(x_2) \right) \\ &= \left(\frac{\pi}{\nu}\right)^{(d-r)/2} \text{vol}(\Lambda_1(\Gamma)) 2^{|\mathbf{k}|} \mathbf{k}! \delta_{\gamma^*, \gamma'^*} \delta_{\mathbf{k}, \mathbf{k}'}. \end{aligned}$$

To conclude, we need to prove completeness. By the uniqueness of the Fourier coefficients in the obtained expansion

$$f(x_1, x_2) = \sum_{\gamma^* \in \Gamma^*, \mathbf{k} \in (\mathbb{Z}^+)^{d-r}} a_{\gamma^*, \mathbf{k}} e^{\frac{\nu}{2} \langle x_1, x_1 \rangle + 2\pi i \langle x_1, v_\chi + \gamma^* \rangle} \mathbf{H}_{\mathbf{k}}^\nu(x_2)$$

for given $f \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$, and the completeness of the Hermite polynomials $(\mathbf{H}_{\mathbf{k}})_\mathbf{k}$ in the Hilbert space $L^2(\mathbb{R}^{d-r}; e^{-\|\xi\|^2} d\lambda)$, we conclude that $a_{\gamma^*, \mathbf{k}} = 0$ for every $\gamma^* \in \Gamma^*$ and $\mathbf{k} \in (\mathbb{Z}^+)^{d-r}$, whenever $\langle f, e_{\gamma^*, \mathbf{k}} \rangle_{\Gamma, \nu} = 0$ for every $\gamma^* \in \Gamma^*$ and $\mathbf{k} \in (\mathbb{Z}^+)^{d-r}$. This implies $f = 0$ and the proof is completed. \square

We conclude this section by determining the completion of $\mathcal{C}_{\Gamma, \chi}^{\nu, \infty}(\mathbb{R}^d)$ with respect to the scalar product (2.3). Namely, we assert

Proposition 3.9. *The completion of $(\mathcal{C}_{\Gamma, \chi}^{\nu, \infty}(\mathbb{R}^d); \|\cdot\|_{\Gamma, \nu})$ coincides with the Hilbert space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$.*

Proof. The completion of $(\mathcal{C}_{\Gamma, \chi}^{\nu, \infty}(\mathbb{R}^d); \|\cdot\|_{\Gamma, \nu})$ is clearly contained in the Hilbert space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. For the converse, notice that the function $f|_{\Lambda(\Gamma)}$ belongs to $L^2(\Lambda(\Gamma); e^{-\nu\|x\|^2} d\lambda)$ whenever $f \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. Thus, there exists a sequence $(\phi_n)_n$ of \mathcal{C}^∞ functions with support included in the interior of $\Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is arbitrary, it follows that ϕ_n is defined on the whole \mathbb{R}^d and furthermore satisfying (3.1). Hence $(\phi_n)_n$ is a Cauchy sequence belonging to $(\mathcal{C}_{\Gamma, \chi}^{\nu, \infty}(\mathbb{R}^d); \|\cdot\|_{\Gamma, \nu})$ and converges to f . This shows that f belongs to the completion of $\mathcal{C}_{\Gamma, \chi}^{\nu, \infty}(\mathbb{R}^d)$. \square

4. ON THETA BARGMANN-FOCK SPACE $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$

In this section, we review briefly some needed properties of the (Γ, χ) -theta Bargmann-Fock space, i.e., the Hilbert $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$ of the (L^2, Γ, χ) -holomorphic theta functions on \mathbb{C}^d . Here, we restrict ourself to the special case of Γ being a discrete subgroup of rank r in $(\mathbb{R}^d, +)$ that can be viewed as a discrete subgroup in $(\mathbb{C}^d, +)$. The general case of Γ being an arbitrary isotropic discrete subgroup is considered and discussed in [9]. In fact, we regard \mathbb{C}^d as the complexify of \mathbb{R}^d , $\mathbb{C}^d = \mathbb{R}^d + i\mathbb{R}^d$, that we endow with the natural extension of the bilinear symmetric form $\langle \cdot, \cdot \rangle$ on \mathbb{R}^d , to wit

$$\langle z, w \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_d v_d, \quad (4.1)$$

for $z = (u_1, u_2, \dots, u_d)$ and $w = (v_1, v_2, \dots, v_d)$ in \mathbb{C}^d , as well as the standard hermitian scalar product

$$H(z, w) = \langle z, \bar{w} \rangle. \quad (4.2)$$

Accordingly, the decomposition $\mathbb{R}^d = \mathbb{V}_\Gamma + \mathbb{V}_\Gamma^{\perp(\cdot)}$ can be extended naturally to

$$\mathbb{C}^d = \mathbb{V}_\mathbb{C} \oplus \mathbb{V}_\mathbb{C}^{\perp_H}, \quad (4.3)$$

where $\mathbb{V}_\mathbb{C}$ and $\mathbb{V}_\mathbb{C}^{\perp_H}$ are the complex subspaces $\mathbb{V}_\mathbb{C} = \mathbb{V}_\Gamma + i\mathbb{V}_\Gamma$ and $\mathbb{V}_\mathbb{C}^{\perp_H} = \mathbb{V}_\Gamma^{\perp(\cdot)} + i\mathbb{V}_\Gamma^{\perp(\cdot)}$. Hence, the considered discrete rank r subgroup Γ is an isotropic subgroup of $(\mathbb{C}^d, +)$, in the sense that $\Im m(H(\gamma, \gamma')) = 0$ for all $\gamma, \gamma' \in \Gamma$.

Within the above notations, $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$ is the space of all holomorphic functions on \mathbb{C}^d displaying the functional equation

$$f(z + \gamma) = \chi(\gamma) e^{\nu H(z + \frac{\gamma}{2}, \gamma)} f(z); \quad z \in \mathbb{C}^d, \gamma \in \Gamma, \quad (4.4)$$

and such that the square norm

$$\|f\|_{\Gamma, H}^2 := \int_{\Lambda(\Gamma)} |f(z)|^2 e^{-\nu H(z, z)} d\lambda(z), \quad (4.5)$$

is finite. Here $\tilde{\Lambda}(\Gamma)$ is a fundamental domain of Γ in $\mathbb{C}^d = \mathbb{R}^{2d}$. The associated hermitian inner scalar product is

$$\langle f, g \rangle_{\Gamma, H} := \int_{\tilde{\Lambda}(\Gamma)} f(z) \overline{g(z)} e^{-\nu H(z, z)} d\lambda(z). \quad (4.6)$$

The needed properties related to $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$ are summarized in the following

Theorem 4.1. *The space $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$ is nontrivial if and only if χ is a character. In this case, the (Γ, χ) -theta Bargmann-Fock space $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$ is a reproducing kernel Hilbert space. Moreover, the set of functions*

$$\varphi_{\gamma^*, \mathbf{k}}(z_1, z_2) = e^{\frac{\nu}{2} \langle z_1, z_1 \rangle + 2\pi i \langle z_1, \gamma^* + v_\chi \rangle} (z_2)^{\mathbf{k}}; \quad z_1 \in \mathbb{V}_{\mathbb{C}}, z_2 \in \mathbb{V}_{\mathbb{C}}^{\perp H}, \quad (4.7)$$

for varying $\gamma^* \in \Gamma^*$ and multi-index $\mathbf{k} \in (\mathbb{Z}^+)^{d-r}$, constitutes an orthogonal basis of $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$ with

$$\|\varphi_{\gamma^*, \mathbf{k}}\|_{\Gamma, H}^2 = \left(\frac{\pi}{\nu}\right)^{d-\frac{r}{2}} \frac{\text{vol}(\Lambda_1(\Gamma))}{2^{r/2}} \frac{k!}{\nu^{|\mathbf{k}|}} e^{2\frac{\pi^2}{\nu} \langle \gamma^* + v_\chi, \gamma^* + v_\chi \rangle}. \quad (4.8)$$

The proofs of the first statements are quite similar to the one provided in [9]. To prove the last one, we make use of the following Lemma.

Lemma 4.2 ([2]). *A Γ -periodic holomorphic function h on $\mathbb{V}_{\mathbb{C}}$ can be expanded as follows*

$$h(w) = \sum_{\gamma^* \in \Gamma^*} b_{\gamma^*}(w) e^{2\pi i \langle w, \gamma^* \rangle},$$

where the series converges absolutely and uniformly on every compact subset of $\mathbb{V}_{\mathbb{C}}$. Furthermore, the coefficients $b_{\gamma^*}(w)$ are given by

$$b_{\gamma^*}(w) = \frac{1}{\text{vol}(\Lambda_1(\Gamma))} \int_{\Lambda_1(\Gamma)} h(w) e^{-2\pi i \langle \gamma^*, w \rangle} d\lambda(x_1); \quad w = x_1 + iy_1, \quad (4.9)$$

and are independents of $\Im m(w)$.

Proofs of (4.7) and (4.8). Associated to a given holomorphic function f displaying (4.4), we consider

$$h(z) := h(z_1, z_2) = e^{-\frac{\nu}{2} \langle z_1, z_1 \rangle - 2\pi i \langle z_1, v_\chi \rangle} f(z_1, z_2). \quad (4.10)$$

Hence, we can easily check that h is a holomorphic Γ -periodic function, and therefore the function $z_1 \mapsto h_{z_2}(z_1) := h(z_1, z_2)$ can be expanded as

$$h_{z_2}(z_1) = \sum_{\gamma^* \in \Gamma^*} b_{\gamma^*}(z_2) e^{2\pi i \langle z_1, \gamma^* \rangle}, \quad (4.11)$$

for fixed z_2 in $\mathbb{V}_{\mathbb{C}}^{\perp H}$, according to Lemma 4.2. The Fourier coefficients $b_{\gamma^*}(z_2)$ are given through (4.9) with $h(z_1, z_2) = h_{z_2}(z_1)$ and they are holomorphic on $\mathbb{V}_{\mathbb{C}}^{\perp H}$. Thus, they can be written as

$$b_{\gamma^*}(z_2) = \sum_{\mathbf{k} \in (\mathbb{Z}^+)^{d-r}} b_{\gamma^*, \mathbf{k}} z_2^{\mathbf{k}},$$

for some complex numbers $b_{\gamma^*, \mathbf{k}}$. Above, z_2 is identified with their coordinates in $\mathbb{V}_{\mathbb{C}}^{\perp H}$ and the functions $\varphi_{\gamma^*, \mathbf{k}}$ are generators of $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$. For the orthogonality, we use Fubini's theorem to get

$$\begin{aligned} \langle \varphi_{\gamma^*, \mathbf{k}}, \varphi_{\gamma'^*, \mathbf{k}'} \rangle_{\Gamma, H} &= \left(\int_{\Lambda_1(\Gamma) \times \mathbb{V}_{\Gamma}} e^{\frac{\nu}{2} (\langle z_1, z_1 \rangle + \langle \bar{z}_1, \bar{z}_1 \rangle - 2H(z_1, z_1)) + 2i\pi \langle z_1 - \bar{z}_1, v_\chi \rangle + 2i\pi (\langle \gamma^*, z_1 \rangle - \langle \gamma'^*, \bar{z}_1 \rangle)} d\lambda(z_1) \right) \\ &\quad \times \left(\int_{\mathbb{C}^{d-r}} z_2^{\mathbf{k}} \bar{z}_2^{\mathbf{k}'} e^{-\nu |z_2|^2} d\lambda(z_2) \right). \end{aligned}$$

In the second integral in the right hand-side of the last identity, we recognize the scalar product of the monomials in the classical Bargmann-Fock space on \mathbb{C}^{d-r} , to wit

$$\int_{\mathbb{C}^{d-r}} z_2^{\mathbf{k}} \bar{z}_2^{\mathbf{k}'} e^{-\nu |z_2|^2} d\lambda(z_2) = \left(\frac{\pi}{\nu}\right)^{d-r} \frac{k!}{\nu^{|\mathbf{k}|}} \delta_{\mathbf{k}, \mathbf{k}'}.$$

For $z_1 = x_1 + iy_1$ with $x_1, y_1 \in \mathbb{V}_\Gamma$, we have $\langle \gamma^*, z_1 \rangle - \langle \gamma^{*'}, \bar{z}_1 \rangle = \langle \gamma^* - \gamma^{*'}, x_1 \rangle + i \langle \gamma^{*'} + \gamma^*, y_1 \rangle$. Note also that since $H(z_1, z_1) = \langle z_1, \bar{z}_1 \rangle$, it follows $\langle z_1, z_1 \rangle + \langle \bar{z}_1, \bar{z}_1 \rangle - 2H(z_1, z_1) = -4 \langle y_1, y_1 \rangle$. Hence, we get

$$\begin{aligned} \langle \varphi_{\gamma^*, \mathbf{k}}, \varphi_{\gamma^{*'}, \mathbf{k}'} \rangle_{\Gamma, H} &= \left(\frac{\pi}{\nu}\right)^{d-r} \frac{k!}{\nu^{|\mathbf{k}|}} \left(\int_{\Lambda_1(\Gamma)} e^{2i\pi \langle \gamma^* - \gamma^{*'}, x_1 \rangle} d\lambda(x_1) \right) \\ &\quad \times \left(\int_{\mathbb{V}_\Gamma} e^{-2 \langle y_1, y_1 \rangle - 2\pi \langle \gamma^{*'} + \gamma^* + 2v_\chi, y_1 \rangle} d\lambda(y_1) \right) \delta_{\mathbf{k}, \mathbf{k}'} \\ &= (\text{vol}(\Lambda_1(\Gamma)))^2 \left(\frac{\pi}{\nu}\right)^{d-r} \frac{k!}{\nu^{|\mathbf{k}|}} \left(\int_{\mathbb{R}^r} e^{-2y_1 G y_1 - 4\pi(\beta_\chi + m) G y_1} d\lambda(y_1) \right) \delta_{\gamma^*, \gamma^{*'}} \delta_{\mathbf{k}, \mathbf{k}'} \end{aligned}$$

To conclude, we need to the following explicit expression for the gaussian integral

Lemma 4.3 ([6]). *Let $a > 0$, $b \in \mathbb{C}^s$ and A a symmetric $s \times s$ matrix whose $\Re(A)$ is positive definite. Then,*

$$\int_{\mathbb{R}^s} e^{-ayAy + by} dy = \left(\frac{1}{\sqrt{\det A}} \right) \left(\frac{\pi}{a} \right)^{s/2} e^{\frac{1}{4a} b A^{-1} b}. \quad (4.12)$$

Hence by means of (4.12), we obtain

$$\langle \varphi_{\gamma^*, \mathbf{k}}, \varphi_{\gamma^{*'}, \mathbf{k}'} \rangle_{\Gamma, H} = \left(\frac{\pi}{\nu}\right)^{d-\frac{r}{2}} \frac{\text{vol}(\Lambda_1(\Gamma))}{2^{r/2}} \frac{k!}{\nu^{|\mathbf{k}|}} e^{2\frac{\pi^2}{\nu} \langle \gamma^* + v_\chi, \gamma^{*'} + v_\chi \rangle} \delta_{\gamma^*, \gamma^{*'}} \delta_{\mathbf{k}, \mathbf{k}'} \quad (4.13)$$

The completeness can be obtained by proceeding in a similar way as in [9]. \square

5. PROOFS OF THEOREMS 2.2 AND 2.3: THE THETA-SEGAL-BARGMANN TRANSFORM

In this section we look for the action of the so-called Segal-Bargmann transform \mathcal{B} on $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$. Recall that for given $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$, we have [1]

$$[\mathcal{B}\varphi](z) = \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} \int_{\mathbb{R}^d} e^{\sqrt{2}\nu \langle z, x \rangle - \frac{\nu}{2} \langle z, z \rangle} \varphi(x) e^{-\nu \|x\|^2} d\lambda(x); \quad z \in \mathbb{C}^d = \mathbb{R}^{2d}, \quad (5.1)$$

provided that the integral exists. Then, it is a well-known fact that this transform maps isometrically the classical space $L^2(\mathbb{R}^d; e^{-\nu \|x\|^2} d\lambda)$, of all $e^{-\nu \|x\|^2} d\lambda$ -square integrable functions on \mathbb{R}^d , onto the classical Bargmann-Fock space $\mathcal{F}^{2, \nu}(\mathbb{C}^d)$. Its kernel function turns out to be the exponential generating function of the Hermite polynomials. Notice that the space $L^2(\mathbb{R}^d; e^{-\nu \|x\|^2} d\lambda)$ (resp. $\mathcal{F}^{2, \nu}(\mathbb{C}^d)$) corresponds to $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ (resp. $\mathcal{F}_{\Gamma, \chi}^{2, \nu}(\mathbb{C}^d)$) with Γ and χ are trivial, $\Gamma = \{0\}$ and $\chi = 1$. In order to characterize $\mathcal{B}(L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d))$ for arbitrary rank r discrete subgroup Γ and character χ , we begin with the following

Proposition 5.1. *The integral operator \mathcal{B} is well defined on $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ and the integral defining $\mathcal{B}\varphi$; for $\varphi \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$, converges uniformly on compact sets of \mathbb{C}^d .*

Proof. Starting from the fact $\mathbb{R}^d = \bigcup_{\gamma \in \Gamma} (\gamma + \Lambda(\Gamma))$, we can rewrite the Segal-Bargmann transform as

$$\begin{aligned} [\mathcal{B}\varphi](z) &= \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} e^{\frac{\nu}{2} \langle z, z \rangle} \int_{\mathbb{R}^d} e^{-\frac{\nu}{2} \langle x - \sqrt{2}z, x - \sqrt{2}z \rangle - \frac{\nu}{2} \langle x, x \rangle} \varphi(x) d\lambda(x) \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} e^{\frac{\nu}{2} \langle z, z \rangle} \sum_{\gamma \in \Gamma} \int_{\gamma + \Lambda(\Gamma)} e^{-\frac{\nu}{2} \langle x - \sqrt{2}z, x - \sqrt{2}z \rangle - \frac{\nu}{2} \langle x, x \rangle} \varphi(x) d\lambda(x). \end{aligned}$$

Replacing x by $x + \gamma$ in the above integral and using (2.1) satisfied by φ , we get

$$[\mathcal{B}\varphi](z) = \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} e^{\frac{\nu}{2}\langle z, z \rangle} \times \left(\sum_{\gamma \in \Gamma} \chi(\gamma) e^{-\frac{\nu}{2}\langle \gamma, \gamma \rangle + \sqrt{2}\nu\langle z, \gamma \rangle} \int_{\Lambda(\Gamma)} e^{-\frac{\nu}{2}\langle x - \sqrt{2}z, x - \sqrt{2}z \rangle - \nu\langle \gamma, x \rangle - \frac{\nu}{2}\langle x, x \rangle} \varphi(x) d\lambda(x) \right) \quad (5.2)$$

and therefore

$$|[\mathcal{B}\varphi](z)| \leq \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} \sum_{\gamma \in \Gamma} |e^{\frac{\nu}{2}\langle z, z \rangle - \frac{\nu}{4}\langle \gamma, \gamma \rangle + \sqrt{2}\nu\langle z, \gamma \rangle} I_{\nu, \Gamma}(\varphi)(z_1, z_2)|, \quad (5.3)$$

since $-\langle \gamma, x \rangle = -\langle \gamma, x_1 \rangle \leq \langle x_1, x_1 \rangle + \frac{1}{4}\langle \gamma, \gamma \rangle$. The quantity $I_{\nu, \Gamma}(\varphi)(z_1, z_2)$ stands for

$$I_{\nu, \Gamma}(\varphi)(z_1, z_2) = \int_{\Lambda_1(\Gamma) \times \mathbf{V}_\Gamma^{\perp(\cdot, \cdot)}} e^{-\frac{\nu}{2}\langle x - \sqrt{2}z, x - \sqrt{2}z \rangle + \frac{\nu}{2}\langle x_1, x_1 \rangle - \frac{\nu}{2}\langle x_2, x_2 \rangle} \varphi(x) d\lambda(x_1) d\lambda(x_2).$$

The Cauchy-Schwarz inequality combined with the Fubini theorem yields the following estimate

$$|I_{\nu, \Gamma}(\varphi)(z_1, z_2)|^2 \leq \|\varphi\|_{\Gamma, \nu}^2 \left(\int_{\Lambda_1(\Gamma)} \left| e^{-\frac{\nu}{2}\langle x_1 - \sqrt{2}z_1, x_1 - \sqrt{2}z_1 \rangle + \frac{\nu}{2}\langle x_1, x_1 \rangle} \right|^2 d\lambda(x_1) \right) \times \left(\int_{\mathbf{V}_\Gamma^{\perp(\cdot, \cdot)}} \left| e^{-\frac{\nu}{2}\langle x_2 - \sqrt{2}z_2, x_2 - \sqrt{2}z_2 \rangle - \frac{\nu}{2}\langle x_2, x_2 \rangle} \right|^2 d\lambda(x_2) \right).$$

The first integral involved in the right hand-side of the previous inequality is clearly finite for $\Lambda_1(\Gamma)$ being compact. The second one can be shown to be finite using Lemma 4.3. Moreover, for every z in any compact set $K \subset \mathbb{C}^d$, we have $\Re(\langle \gamma, z \rangle) \leq \frac{1}{8}\langle \gamma, \gamma \rangle$ for $\gamma \in \Gamma$ outside certain disc $D(0, R)$; $R > 0$. Thus, there exists a constant c_K such that

$$|[\mathcal{B}\varphi](z)| \leq c_K \sum_{\gamma \in \Gamma} e^{-\frac{\nu}{8}\langle \gamma, \gamma \rangle}. \quad (5.4)$$

This means that the quantity $\mathcal{B}\varphi$ is well defined for every $\varphi \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$, since the series in the right hand-side of (5.2) converges. \square

A part of the proof of Theorem (2.2) is contained in the following proposition showing the image of the (L^2, Γ, χ) -likewise theta functions on \mathbb{R}^d , by the \mathcal{B} , are the $(\tilde{\Gamma}, \tilde{\chi})$ -holomorphic theta functions on \mathbb{C}^d , where $\tilde{\Gamma}$ is the scaled discrete subgroup $\tilde{\Gamma} := \Gamma/\sqrt{2}$ and $\tilde{\chi}$ is the $\tilde{\Gamma}$ -character defined by

$$\tilde{\chi}(\tilde{\gamma}) = e^{2i\pi\langle \tilde{\gamma}, \sqrt{2}v_\chi \rangle}.$$

Proposition 5.2. *For every $\varphi \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$, the function $z \mapsto [\mathcal{B}\varphi](z)$ is holomorphic on \mathbb{C}^d and satisfies the functional equation*

$$[\mathcal{B}\varphi](z + \tilde{\gamma}) = \tilde{\chi}(\tilde{\gamma}) e^{\nu H(z + \frac{\tilde{\gamma}}{2}, \tilde{\gamma})} [\mathcal{B}\varphi](z) \quad (5.5)$$

for every z in \mathbb{C}^d and every $\tilde{\gamma}$ in $\tilde{\Gamma}$.

Proof. For given $\varphi \in L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$, the function $z \mapsto e^{\sqrt{2}\nu\langle z, x \rangle - \frac{\nu}{2}\langle z, z \rangle - \nu\|x\|^2} \varphi(x)$ involved in the integrand of $\mathcal{B}\varphi$ is clearly holomorphic in z for every fixed $x \in \mathbb{R}^d$. Thus, by the uniform convergence of the integral in $\mathcal{B}\varphi$ on compact subsets of \mathbb{C}^d (Proposition 5.2), it follows that $[\mathcal{B}\varphi](z)$ is holomorphic on \mathbb{C}^d . Moreover, direct computation infers

$$[\mathcal{B}\varphi](z + \frac{1}{\sqrt{2}}\gamma) = \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} e^{\frac{\nu}{2}\langle z, z \rangle + \nu\langle z + \frac{\tilde{\gamma}}{2}, \tilde{\gamma} \rangle} \int_{\mathbb{R}^d} e^{-\frac{\nu}{2}\langle x - \sqrt{2}z - \gamma, x - \sqrt{2}z - \gamma \rangle - \frac{\nu}{2}\|x\|^2} \varphi(x) d\lambda(x).$$

Now, making use of the change $y = x - \gamma$ as well as the fact that φ satisfies the functional equation (2.1), we obtain

$$[\mathcal{B}\varphi](z + \tilde{\gamma}) = \chi(\gamma)e^{\nu\langle z + \frac{\tilde{\gamma}}{2}, \tilde{\gamma} \rangle} [\mathcal{B}\varphi](z) = \tilde{\chi}(\tilde{\gamma})e^{\nu H(z + \frac{\tilde{\gamma}}{2}, \tilde{\gamma})} [\mathcal{B}\varphi](z),$$

since $\langle z + \frac{\tilde{\gamma}}{2}, \tilde{\gamma} \rangle = H(z + \frac{\tilde{\gamma}}{2}, \tilde{\gamma})$ for $\tilde{\gamma} = \tilde{\gamma}$ and $\chi(\gamma) = e^{2i\pi\langle \gamma, v_\chi \rangle} = e^{2i\pi\langle \frac{\gamma}{\sqrt{2}}, \sqrt{2}v_\chi \rangle} = \tilde{\chi}(\tilde{\gamma})$ by Lemma 3.5. \square

The proof of Theorem 2.2, i.e., $\mathcal{B}(L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)) = \mathcal{F}_{\tilde{\Gamma}, \tilde{\chi}}^{2, \nu}(\mathbb{C}^d)$ isometrically, reduces further to show that the Segal-Bargmann transform \mathcal{B} maps an orthonormal basis $e_{\gamma^*, \mathbf{k}}$ of the Hilbert space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ to the an orthonormal one of the Hilbert space $\mathcal{F}_{\tilde{\Gamma}, \tilde{\chi}}^{2, \nu}(\mathbb{C}^d)$.

Proof of Theorem 2.2. Notice first that the action of \mathcal{B} on the basis $e_{\gamma^*, \mathbf{k}}$ of the Hilbert space $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ is given by

$$\begin{aligned} [\mathcal{B}e_{\gamma^*, \mathbf{k}}](z) &= \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} e^{-\frac{\nu}{2}\langle z_1, z_1 \rangle} \left(\int_{\mathbb{V}_\Gamma} e^{\sqrt{2}\nu\langle x_1, z_1 \rangle + 2i\pi\langle x_1, \gamma^* + v_\chi \rangle - \frac{\nu}{2}\langle x_1, x_1 \rangle} d\lambda(x_1) \right) \\ &\quad \times \left(\int_{\mathbb{V}_\Gamma^\perp \langle \cdot, \cdot \rangle} e^{-\frac{\nu}{2}\langle z_2, z_2 \rangle + \sqrt{2}\nu\langle x_2, z_2 \rangle} \mathbf{H}_{\mathbf{k}}^\nu(x_2) e^{-\nu\langle x_2, x_2 \rangle} d\lambda(x_2) \right). \end{aligned} \quad (5.6)$$

In the right hand-side of the second integral occurring in (5.6), we recognize the action of the Segal-Bargmann transform of the Hermite polynomial on \mathbb{R}^{d-r} given by

$$\left(\frac{\nu}{\pi}\right)^{\frac{3(d-r)}{4}} \int_{\mathbb{V}_\Gamma^\perp \langle \cdot, \cdot \rangle} e^{-\frac{\nu}{2}\langle z_2, z_2 \rangle + \sqrt{2}\nu\langle x_2, z_2 \rangle} \mathbf{H}_{\mathbf{k}}^\nu(x_2) e^{-\nu\langle x_2, x_2 \rangle} d\lambda(x_2) = \left(\frac{\nu}{\pi}\right)^{\frac{d-r}{4}} (2\nu)^{|\mathbf{k}|/2} z_2^{\mathbf{k}}. \quad (5.7)$$

The first integral in the right hand-side of (5.6) can be handled by applying Lemma 4.3. Indeed, if $G := (\langle \omega_j, \omega_k \rangle)_{j,k=1}^r$ denotes the Gram-Schmidt matrix of $Q_1 = \langle \cdot, \cdot \rangle$ on \mathbb{V}_Γ with respect to the basis $\omega_1, \dots, \omega_r$, and by $x_1 \in \mathbb{R}^r$, $z_1 \in \mathbb{C}^r$, $m^* \in \mathbb{R}^r$ and $\beta_\chi \in \mathbb{R}^r$ the coordinates of $x_1 \in \mathbb{V}_\Gamma$, $z_1 \in \mathbb{V}_\mathbb{C}$, γ^* and v_χ , respectively, we get

$$\begin{aligned} &\int_{\mathbb{V}_\Gamma} e^{\sqrt{2}\nu\langle x_1, z_1 \rangle + 2i\pi\langle x_1, \gamma^* + v_\chi \rangle - \frac{\nu}{2}\langle x_1, x_1 \rangle} d\lambda(x_1) \\ &= \text{vol}(\Lambda_1(\Gamma)) \int_{\mathbb{R}^r} e^{-\frac{\nu}{2}x_1 G x_1 + x_1 G (\sqrt{2}\nu z_1 + 2i\pi(m^* + \beta_\chi))} d\lambda(x_1) \\ &= \left(\frac{2\pi}{\nu}\right)^{\frac{r}{2}} e^{\nu\langle z_1, z_1 \rangle + 2i\pi\sqrt{2}\langle z_1, \gamma^* + v_\chi \rangle - 2\frac{\pi^2}{\nu}\langle \gamma^* + v_\chi, \gamma^* + v_\chi \rangle} \\ &\stackrel{\text{Lemma 4.3}}{=} \left(\frac{2\pi}{\nu}\right)^{\frac{r}{2}} e^{\nu\langle z_1, z_1 \rangle + 2i\pi\langle z_1, \tilde{\gamma}^* + \tilde{v}_\chi \rangle - \frac{\pi^2}{\nu}\langle \tilde{\gamma}^* + \tilde{v}_\chi, \tilde{\gamma}^* + \tilde{v}_\chi \rangle}. \end{aligned} \quad (5.8)$$

Finally, by taking into account the fact that $\tilde{\Gamma}^* = \sqrt{2}\Gamma^*$ and inserting (5.8) and (5.7) in (5.6), one obtains $[\mathcal{B}e_{\gamma^*, \mathbf{k}}](z) = \varphi_{\tilde{\gamma}^*, \mathbf{k}}$. This completes the proof, since $\varphi_{\tilde{\gamma}^*, \mathbf{k}}$, for varying $\tilde{\gamma}^* \in \tilde{\Gamma}^*$ and $\mathbf{k} \in (\mathbb{Z}^+)^{d-r}$, constitute a complete orthogonal system of $\mathcal{F}_{\tilde{\Gamma}, \tilde{\chi}}^{2, \nu}(\mathbb{C}^d)$. \square

Below, we give a proof of Theorem 2.3 saying that the Segal-Bargmann transform on $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ can be realized as

$$[\mathcal{B}\varphi](z) = \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} \int_{\Lambda(\Gamma)} e^{\sqrt{2}\nu\langle z, x \rangle - \frac{\nu}{2}\langle z, z \rangle} \Theta_{0, G\beta_\chi} \left(\frac{i\nu}{2\pi} G(x_1 - \sqrt{2}z_1) \middle| \frac{i\nu}{2\pi} G \right) \varphi(x) e^{-\nu\|x\|^2} d\lambda(x), \quad (5.9)$$

where $\beta_\chi = (\beta_1, \dots, \beta_r) \in \mathbb{R}^r$ are the coordinates of v_χ in the basis $\omega_1, \dots, \omega_r$.

Proof of Theorem 2.3. Starting from (5.2), we need to give a closed expression of the sum in the right hand-side in terms of the modified theta function. In fact, for $\gamma \in \Gamma$ with the coordinates $m \in \mathbb{Z}^r$, we have

$$\begin{aligned} \sum_{\gamma \in \Gamma} \chi(\gamma) e^{-\frac{\nu}{2} \langle \gamma, \gamma \rangle + \nu \langle \sqrt{2}z - x, \gamma \rangle} &= \sum_{\gamma \in \Gamma} e^{-\frac{\nu}{2} \langle \gamma, \gamma \rangle + \nu \langle \sqrt{2}z_1 - x_1 + \frac{2i\pi}{\nu} v_\chi, \gamma \rangle} \\ &= \sum_{m \in \mathbb{Z}^r} e^{-\frac{\nu}{2} m G m + \nu m G (\sqrt{2}z_1 - x_1 + \frac{2i\pi}{\nu} \beta_\chi)} \\ &= \Theta_{0, G\beta_\chi} \left(\frac{iv}{2\pi} G(x_1 - \sqrt{2}z_1) \middle| \frac{iv}{2\pi} G \right). \end{aligned}$$

The vectors x_1 and z_1 are identified with their coordinates relatively to the basis $\omega_1, \dots, \omega_r$ as mentioned in proof of Lemma 3.5. \square

We conclude this paper by noting that the kernel function

$$A_{\Gamma, \chi}^\nu(z; x) := \left(\frac{\nu}{\pi} \right)^{\frac{3d}{4}} e^{\sqrt{2}\nu \langle z, x \rangle - \frac{\nu}{2} \langle z, z \rangle} \Theta_{0, G\beta_\chi} \left(\frac{iv}{2\pi} G(x_1 - \sqrt{2}z_1) \middle| \frac{iv}{2\pi} G \right), \quad (5.10)$$

is in fact the bilateral generating function corresponding to the orthogonal basis $e_{\gamma^*, \mathbf{k}}$ of $L_{\Gamma, \chi}^{2, \nu}(\mathbb{R}^d)$ and $\varphi_{\sqrt{2}\gamma^*, \mathbf{k}}$ of $\mathcal{F}_{\tilde{\Gamma}, \tilde{\chi}}^{2, \nu}(\mathbb{C}^d)$. Namely, we assert the following

Proposition 5.3. *We have*

$$A_{\Gamma, \chi}^\nu(z; x) = \sum_{\gamma^* \in \Gamma^*, \mathbf{k} \in (\mathbb{Z}^+)^{d-r}} \frac{\varphi_{\sqrt{2}\gamma^*, \mathbf{k}}(z)}{\|\varphi_{\sqrt{2}\gamma^*, \mathbf{k}}\|_{\tilde{\Gamma}, H}} \frac{\overline{e_{\gamma^*, \mathbf{k}}(x)}}{\|e_{\gamma^*, \mathbf{k}}\|_{\Gamma, \nu}}. \quad (5.11)$$

Proof. Using the explicit expressions of $\varphi_{\sqrt{2}\gamma^*, \mathbf{k}}$ given by (4.7) and of $e_{\gamma^*, \mathbf{k}}$ given by (3.18) as well as their norms given respectively through (4.8) and (3.20), we can rewrite the right hand-side of (5.11) as

$$\begin{aligned} &\frac{2^{r/2}}{\text{vol}(\Lambda_1(\Gamma))} \left(\frac{\nu}{\pi} \right)^{\frac{3d}{4} - \frac{r}{2}} e^{\frac{\nu}{2} (\langle z_1, z_1 \rangle + \langle x_1, x_1 \rangle)} \\ &\times \sum_{\gamma^* \in \Gamma^*, \mathbf{k} \in (\mathbb{Z}^+)^{d-r}} e^{-\frac{\pi^2}{\nu} \langle \sqrt{2}\gamma^* + \sqrt{2}v_\chi, \sqrt{2}\gamma^* + \sqrt{2}v_\chi \rangle + 2\pi i \langle \sqrt{2}z_1 - x_1, \gamma^* + v_\chi \rangle} \frac{(\sqrt{\frac{\nu}{2}} z_2)^{\mathbf{k}} \mathbf{H}_{\mathbf{k}}^\nu(x_2)}{\mathbf{k}!}. \end{aligned}$$

Using the generating formula for Hermite polynomials (see for example [12, p. 60]), we get

$$\sum_{\mathbf{k} \in (\mathbb{Z}^+)^{d-r}} \frac{(\sqrt{\frac{\nu}{2}} z_2)^{\mathbf{k}} \mathbf{H}_{\mathbf{k}}^\nu(x_2)}{\mathbf{k}!} = e^{-\frac{\nu}{2} \langle z_2, z_2 \rangle + \sqrt{2}\nu \langle x_2, z_2 \rangle}.$$

Now, note that the dual lattice Γ^* can be identified with $G^{-1}\Gamma$, and hence

$$\begin{aligned} &\sum_{\gamma^* \in \Gamma^*} e^{-\frac{\pi^2}{\nu} \langle \sqrt{2}\gamma^* + \sqrt{2}v_\chi, \sqrt{2}\gamma^* + \sqrt{2}v_\chi \rangle + 2\pi i \langle \sqrt{2}z_1 - x_1, \gamma^* + v_\chi \rangle} \\ &= e^{-\frac{2\pi^2}{\nu} \langle v_\chi, v_\chi \rangle + 2i\pi \langle \sqrt{2}z_1 - x_1, v_\chi \rangle} \sum_{m \in \mathbb{Z}^r} e^{-\frac{2\pi^2}{\nu} m G^{-1} m + 2i\pi (\sqrt{2}z_1 - x_1 + \frac{2i\pi}{\nu} \beta_\chi) m} \\ &= e^{-\frac{2\pi^2}{\nu} \langle v_\chi, v_\chi \rangle + 2i\pi \langle \sqrt{2}z_1 - x_1, v_\chi \rangle} \Theta_{0,0} \left(\sqrt{2}z_1 - x_1 + \frac{2i\pi}{\nu} \beta_\chi \middle| - \left(\frac{iv}{2\pi} G \right)^{-1} \right). \end{aligned}$$

Thus, in view of the well-known identity satisfied by the theta function [13, p. 195]

$$\Theta_{0,0}(\Omega^{-1}z \mid -\Omega^{-1}) = \sqrt{\det(-i\Omega)} e^{i\pi z \Omega^{-1} z} \Theta_{0,0}(z \mid \Omega), \quad (5.12)$$

with $\Omega = \frac{iv}{2\pi}G$, one can see that the left hand-side in (5.11) reduces further to

$$\left(\frac{v}{\pi}\right)^{\frac{3d}{4}} e^{\sqrt{2}v\langle z,x\rangle - \frac{v}{2}\langle z,z\rangle} \Theta_{0,G\beta_\chi} \left(\frac{iv}{2\pi}G(x_1 - \sqrt{2}z_1) \middle| \frac{iv}{2\pi}G\right) = A_{\Gamma,\chi}^v(z;x).$$

□

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